Completing the Square

Introduction:

The procedure called **completing the square** is used to solve quadratic equations
The general form of a quadratic is

$$ax^2 + bx + c = 0$$

Some quadratic expressions are **perfect squares**. For example $x^2 - 6x + 9 = 0$ can be written as

giving $(x-3)^2 = 0$ the solution $x = 3$

Similarly, $4x^2 - 10x + 25$ can be written as $(2x - 5)^2$ which has the solution $x = 2\frac{1}{2}$

In general a quadratic expression is not a perfect square by by adding a number it can become a perfect square and so a solution can be found. This procedure is known as **completing the square** while normally given as an algebraic question it really one of simple arithmetic.

If we consider the equation $x^2 - 6x + 8 = 0$ it is similar to our original example less 1, so it takes little to deduce that to complete the square all you have to do is add 1 to it. At the same time to keep the equation the same 1 must also be added the side, RHS so

$$x^2 - 6x + 8 + 1 = 1 \Rightarrow x^2 - 6x + 9 = 1 \Rightarrow (x-3)^2 = 1$$

Now as before taking the square root of both sides and note that both +1 and -1 squared are 1

$$x - 3 = \pm 1 \Rightarrow x = 4 \text{ or } x = 2$$

This equation was of course easily factorised which would normally be the way to solve a simple equation of this sort.

Consider the equation $2x^2 + 8x - 21 = 0$ it is not so obvious what number needs to be added.

The procedure to find this is straight forward. First dived the whole equation by the coefficient of the x squared and then add half the coefficient in x squared to both the RHS and LHS.

$$x^2 + 4x = \frac{21}{2} \qquad +\left(\frac{4}{2}\right)^2 \qquad \text{half of 4 squared}$$

$$x^2 + 4x + \left(\frac{4}{2}\right)^2 = \frac{21}{2} + \left(\frac{4}{2}\right)^2 \Rightarrow x^2 + 4x + 4 = 4 + \frac{21}{2} \Rightarrow (x+2)^2 = \frac{29}{2}$$

Calculate the terms and gather those not involving x on the RHS. This now gives the squared expression so it then remains to take the square root of each side and calculate the values of x

$$x + 2 = \pm\sqrt{\frac{29}{2}} \Rightarrow x = -2 \pm 3.81 \qquad \text{giving to 2dp } x = -5.81 \text{ or } x = 1.81$$

Of these three equations

$$3x^2 + 8x + 5 = 0 \qquad 2x^2 - 3x - 17 = 0 \qquad 3.5x^2 + 6.5 - 45 = 0$$

The number to add to **complete the squares** are respectively, check why

$$\left(\frac{8}{6}\right)^2 \qquad \left(\frac{3}{4}\right)^2 \qquad \left(\frac{6.5}{7}\right)^2$$

Considering the general equation of the quadratic by completing the square the formula for the solution of quadratics can be derived.

$$ax^2 + bx + c = 0$$

$\Rightarrow \quad x^2 + \frac{b}{a} = -\frac{c}{a}$ The number to add to complete the square is $\left(\frac{b}{2a}\right)^2 \Rightarrow \frac{b^2}{4a^2}$

$$\therefore \Rightarrow \quad x^2 + \frac{b}{a} + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} \Rightarrow \left(x + \frac{b}{2a}\right)^2 = \left(\frac{b^2 - 4ac}{4a^2}\right)$$

Now take the square root of each side remembering the root can be both negative and positive

$$\therefore x + \frac{b}{2a} = \pm\sqrt{\frac{\left(b^2 - 4ac\right)}{4a^2}} \Rightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{\left(b^2 - 4ac\right)}}{2a} \Rightarrow x = \frac{-b \pm \sqrt{\left(b^2 - 4ac\right)}}{2a}$$

Note that if $4ac > b^2$ then there are no real roots.

Questions: Solve by completing the square.

1.) $x^2 + 8x + 7 = 0$
2.) $x^2 + 10x + 3 = 0$
3.) $x^2 - 8x - 2 = 0$
4.) $2x^2 + 8x + 2 = 0$
5.) $3x^2 - 6x + 1 = 0$
6.) $4x^2 - 2x - 5 = 0$
7.) $4x^2 - x = 8$
8.) $2x^2 + k + k^2 = 0$