Integration by Parts

Integration by parts is an integration technique which is based on the Product Rule for derivatives.

From the Product Rule $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

By integrating both sides $\int \frac{d(uv)}{dx}dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$

$$\therefore \int d(uv) = \int udv + \int vdu \quad \Rightarrow \quad uv = \int udv + \int vdu \quad \Rightarrow \quad \int udv = uv - \int vdu$$

Which is the general **integration by parts** formula. Some functions do not have an immediate recognisable integral and it is for these **integration by parts** might be a solution.

If we compare $\int -cos(x)dx$ and $\int 2xsin(x)dx$ we can see immediately that the first integral is $sin(x) + \text{C}$

If you can't then go back and learn the standard integrals.

But there is no apparent solution to the second integral. To use **integration by parts** break the function in to two parts one that can be differentiated and one that can be easily integrated.

$$\int \mathbf{2}x\ sin(x)dx \qquad u = \mathbf{2}x \quad \frac{du}{dx} = \mathbf{2} \qquad \frac{dv}{dx} = sin(x) \quad v = -cos(x)$$

Then substituting into the formula $\int udv = uv - \int vdu$

$$\Rightarrow \quad = 2x\,(-cos(x))dx - \int(-cos(x)2dx$$

$$\Rightarrow \quad = -2xcos(x) + \int 2cos(x)dx \quad \Rightarrow \quad -2xcos(x) + 2sin(x) + \text{C}$$

For the integral $\int xe^x dx \qquad u = x \quad \frac{du}{dx} = e^x dx \qquad \frac{dv}{dx} = \mathbf{1} \quad v = e^x$

$$\Rightarrow \quad xe^x - \int e^x dx \quad \Rightarrow \quad xe^x - e^x + \text{C}$$

Note the constant of integration **C** is ignored in the intermediate steps but must be added at the end.

The method can also be used on functions that appear to only have one part such as $\int ln(x)dx$

$$u = ln(x) \quad \frac{du}{dx} = \frac{\mathbf{1}}{x}dx \qquad \frac{dv}{dx} = \mathbf{1} \quad v = x$$

$$\Rightarrow \quad xln(x) - dx \quad \Rightarrow \quad xln(x) - x + \text{C}$$

As an exercise differentiate the two solutions given and see if they give the original function.

For the integral $\mathbf{I_1} = \quad e^{2x} cos(x)dx$

then let $u = e^{2x} \quad dv = sin(x) \quad \Rightarrow \quad du = 2x^{2x} \quad v = sin(x)$

$$I_1 = e^{2x} sin(x) - 2\int e^{2x} sin(x)$$

which does not give a standard integral so the remaining integral can its self be integrated by parts.

$\mathbf{I_2}= \quad \int e^{2x} sin(x)dx \quad u = e^{2x} \quad dv = sin(x) \quad \Rightarrow \quad du = 2x^{2x} \quad v = -cos(x)$

$\Rightarrow \quad -e^{2x} cos(x) - 2\int e^{2x} cos(x)dx$ which gives the original integral.

$\Rightarrow \quad -e^{2x} cos(x) - 2I_1$ but $I_1 = e^{2x} sin(x) - 2I_2$

Substitute back into $\mathbf{I_1}$

$\mathbf{I_1}= \quad \Rightarrow \quad -e^{2x} cos(x) - 2(-e^{2x} cos(x) - 2I_1$

$\Rightarrow \quad e^{2x} sin(x) + 2e^{2x} cos(x) + 4I_1$

Collecting $\mathbf{I_1}$ to the RHS

$$\Rightarrow \quad 5I_1 = e^{2x} sin(x) + 2e^{2x} cos(x) \quad \Rightarrow \quad I_1 = \frac{e^{2x} sin(x) + 2e^{2x} cos(x)}{5}$$

This example demonstrates the the integration by parts can have more then one step, theoretically there is no limit a tem can be successively differentiated till it disappears. However for such problems there is a more convenient way to handle the solutions by drawing up a table of successive differentiations and integrations. This is covered on different sheets.

It also shows that the original integral can reappear with additional terms. This allows the gathering up of the original integral for a solution.

Questions:

1.) $\int xe^x dx$

2.) $\int x\,sin(x)dx$

3.) $\int x\,ln(x)dx$

4.) $\int e^x cos(x)dx$

5.) $\int xe^{2x} dx$

6.) $\int x^7 ln(3x)dx$

7.) $\int cos(2x)e^{3x} dx$

8.) $\int arctan(x)dx$